IIT-JEE 2009

ESTD. 1993
MASTERMIND CLASSES
MMC
TOTAL QUALITY EFFORTS IN TEACHING

MATHEMATICS

PART – II
SECTION – I
Single Correct Choice Type

This section contains 4 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONLY ONE** is correct.

20. The locus of the orthocentre of the triangle formed by the lines $(1+p)x - py + p(1+p) = 0$, $(1+q)x - qy + q(1+q) = 0$ and $y = 0$, where $p \neq q$ is

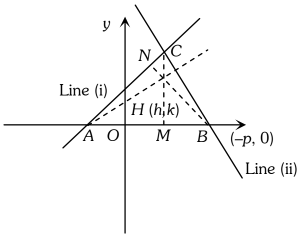
(A) A hyperbola (B) A parabola (C) An ellipse (D) A straight line

Sol. **(D)** $(1+p)x - py + p(1+p) = 0$ .....(i)

$(1+q)x - qy + q(1+q) = 0$ .....(ii)

On solving (i) and (ii), we get $C(pq, (1+p)(1+q))$

$\therefore$ Equation of altitude CM passing through C and perpendicular to AB is $x = pq$ .....(iii)



$\because$ Slope of line (ii) is $= \left(\dfrac{1+q}{q}\right)$

$\therefore$ Slope of altitude BN (as shown in figure) is $= \dfrac{-q}{1+q}$

$\therefore$ Equation of BN is $y - 0 = \dfrac{-q}{1+q}(x+p) \Rightarrow y = \dfrac{-q}{(1+q)}(x+p)$ .....(iv)

Let orthocentre of triangle be $H(h, k)$ which is the point of intersection of (iii) and (iv).

$\therefore$ On solving (iii) and (iv), we get $x = pq$ and $y = -pq \Rightarrow h = pq$ and $k = -pq$

$\therefore$ $h + k = 0$

$\therefore$ Locus of $H(h, k)$ is $x + y = 0$.

21. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x–axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

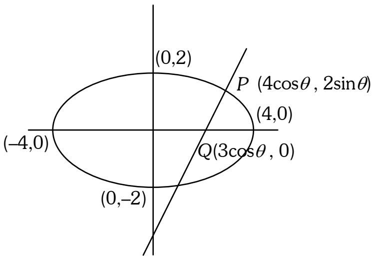
(A) $\left(\pm\dfrac{3\sqrt{5}}{2}, \pm\dfrac{2}{7}\right)$ (B) $\left(\pm\dfrac{3\sqrt{5}}{2}, \pm\dfrac{\sqrt{19}}{4}\right)$ (C) $\left(\pm 2\sqrt{3}, \pm\dfrac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\dfrac{4\sqrt{3}}{7}\right)$

Sol. **(C)** $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

$a = 4,\ b = 2$

Equation of normal $4x\sec\theta - 2y\,\text{cosec}\,\theta = 12$

$M\left(\dfrac{7\cos\theta}{2}, \sin\theta\right) = (h, k)$ (say)



$h = \dfrac{7\cos\theta}{2} \Rightarrow \cos\theta = \dfrac{2h}{7}$ and $k = \sin\theta$

$\dfrac{4h^2}{49} + k^2 = 1$

Locus $\dfrac{4x^2}{49} + y^2 = 1$ .....(i)

For given ellipse $e^2 = 1 - \dfrac{4}{16} = \dfrac{3}{4} \Rightarrow e = \dfrac{\sqrt{3}}{2}$

$x = \pm 4 \times \dfrac{\sqrt{3}}{2} = \pm 2\sqrt{3}$ .....(ii)

Solving (i) and (ii), we get

$\dfrac{4}{49} \times 12 + y^2 = 1 \Rightarrow y^2 = 1 - \dfrac{48}{49} = \dfrac{1}{49} \Rightarrow y = \pm\dfrac{1}{7}$

$\therefore$ Required points $\left(\pm 2\sqrt{3}, \pm\dfrac{1}{7}\right)$.

22. If the sum of first n terms of an A.P. in cn^2, then the sum of squares of these n terms is

(A) $\dfrac{n(4n^2-1)c^2}{6}$ (B) $\dfrac{n(4n^2+1)c^2}{3}$ (C) $\dfrac{n(4n^2-1)c^2}{3}$ (D) $\dfrac{n(4n^2+1)c^2}{6}$

Sol. **(C)** $S_n = cn^2$

$S_{n-1} = c(n-1)^2 = cn^2 + c - 2cn$

$T_n = 2cn - c$

$T_n^2 = (2cn-c)^2 = 4c^2 n^2 + c^2 - 4c^2 n$

Sum $= \Sigma T_n^2 = \dfrac{4c^2 . n(n+1)(2n+1)}{6} + nc^2 - 2c^2 n(n+1)$

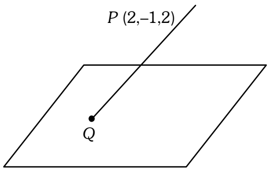
$= \dfrac{2c^2 n(n+1)(2n+1) + 3nc^2 - 6c^2 n(n+1)}{3} = \dfrac{nc^2[4n^2+6n+2+3-6n-6]}{3} = \dfrac{nc^2(4n^2-1)}{3}$.

23. A line with positive direction cosines passes through the point $P(2,-1,2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x+y+z=9$ at point Q. The length of the line segment PQ equals

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

Sol. **(C)** $l = m = n = \dfrac{1}{\sqrt{3}}$

$2x+y+2=9$



$\therefore$ Equations of line are $\dfrac{x-2}{1/\sqrt{3}} = \dfrac{y+1}{1/\sqrt{3}} = \dfrac{z-2}{1/\sqrt{3}}$

$x-2 = y+1 = z-2 = r$

$Q \equiv (r+2, r-1, r+2)$

$\because$ Q lies on the plane $2x+y+z=9$

$2(r+2)+(r-1)+(r+2)=9 \Rightarrow 4r+5=9 \Rightarrow r=1$

$\therefore$ $Q(3,0,3)$

$\therefore$ $PQ = \sqrt{1+1+1} = \sqrt{3}$.

SECTION – II

Multiple Correct Choice Type

This section contains 5 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) for its answer, out of which **ONE OR MORE** is/are correct.

24. For the function $f(x) = x\cos\dfrac{1}{x}, \quad x \ge 1$

(A) For at least one x in the interval $[1,\infty)$, $f(x+2) - f(x) < 2$

(B) $\lim\limits_{x\to\infty} f'(x) = 1$

(C) For all x in the interval $[1,\infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

Sol. **(B,C,D)** $f(x) = x\cos\dfrac{1}{x},\ x \ge 1 \Rightarrow f'(x) = \dfrac{1}{x}\sin\dfrac{1}{x} + \cos\dfrac{1}{x} \Rightarrow f''(x) = -\dfrac{1}{x^3}\cos\left(\dfrac{1}{x}\right)$

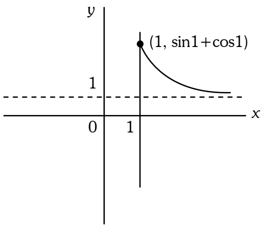
Now, $\lim\limits_{x\to 0} f'(x) = 0 + 1 = 1 \Rightarrow$ option B is correct

$x \in [1,\infty) \Rightarrow \dfrac{1}{x} \in (0,1] \Rightarrow f''(x) < 0 \Rightarrow$ option D is correct.

As $f'(1) = \sin 1 + \cos 1 > 1$

$f'(x)$ is strictly decreasing and $\lim\limits_{x\to\infty} f'(x) = 1$

So graph of $f'(x)$ is as below.



Now, in $[x, x+2]$, $x \in [1,\infty)$, $f(x)$ is continuous and differentiable

So by LMVT, $f'(x) = \dfrac{f(x+2)-f(x)}{2}$ as $f'(x) > 1$ for all $x \in [1,\infty)$ $\Rightarrow \dfrac{f(x+2)-f(x)}{2} > 1 \Rightarrow f(x+2) - f(x) > 2$ for all $x \in [1,\infty)$.

25. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

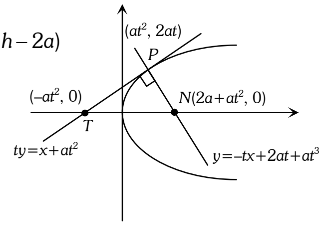
(A) Vertex is $\left(\dfrac{2a}{3}, 0\right)$ (B) Directrix is $x = 0$ (C) Latus rectum is $\dfrac{2a}{3}$ (D) Focus is $(a, 0)$

Sol. **(A,D)** Let centroid of ΔPTN is $R(h, K)$.

$$\therefore\ h = \frac{at^2 + (-at^2) + 2a + at^2}{3} \text{ and } k = \frac{2at}{3} \quad \Rightarrow 3h = 2a + a\,.\left(\frac{3k}{2a}\right)^2 \Rightarrow 3h = 2a + \frac{9k^2}{4a} \Rightarrow 9k^2 = 4a(3h - 2a)$$

Locus of centroid is $y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

$\therefore$ Vertex $\left(\frac{2a}{3}, 0\right)$; directrix $x - \frac{2a}{3} = -\frac{a}{3} \Rightarrow x = \frac{a}{3}$; latus rectum $= \frac{4a}{3}$; focus $\left(\frac{a}{3} + \frac{2a}{3}, 0\right)$, *i.e.*, $(a, 0)$.



26. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx,\ \ n = 0, 1, 2, \ldots\ldots.,$ then

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$ (C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

Sol. **(A,B,C)**

(A) $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx$

$I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\,dx$ [By property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$]

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}\,dx \Rightarrow 2I_n = 2\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx \Rightarrow I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$$

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}\,dx = \int_0^{\pi} \frac{2\cos(n+1)x \sin x}{\sin x}\,dx = 2\left[\frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = 0 \qquad \Rightarrow I_{n+2} = I_n.$$

(B) $I_3 = I_5 = \ldots\ldots. = I_{21}$

$$\therefore \sum_{m=1}^{10} I_{2m+1} = 10 I_3 = 10\int_0^{\pi} \frac{\sin 3x}{\sin x}\,dx = 10\int_0^{\pi} (3 - 4\sin^2 x)\,dx = 10\,[3x - 2x + 2\sin 2x]_0^{\pi} = 10\pi.$$

(C) $I_2 = I_4 = \ldots\ldots. = I_{20}$

$$\sum_{m=1}^{10} I_{2m} = 10\int_0^{\pi} \frac{\sin 2x}{\sin x}\,dx = 20\,[\sin x]_0^{\pi} = 0.$$

27. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is(are)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

Sol. **(C,D)** $0 < \theta < \frac{\pi}{2}$, $\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$

$$\Rightarrow \sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2} \Rightarrow \sum_{m=1}^{6} \frac{\sin\left[\theta + \frac{m\pi}{4} - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\frac{\pi}{4}\left\{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)\right\}} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)}{\frac{1}{\sqrt{2}}} = 4\sqrt{2} \Rightarrow \sum_{m=1}^{6} \cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right) = 4$$

$$\Rightarrow \cot(\theta) - \cot\left(\theta + \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) - \cot\left(\theta + \frac{2\pi}{4}\right) + \ldots.. + \cot\left(\theta + \frac{5\pi}{4}\right) - \cot\left(\theta + \frac{6\pi}{4}\right) = 4$$

$$\Rightarrow \cot\theta - \cot\left(\frac{3\pi}{2} + \theta\right) = 4 \Rightarrow \cot\theta + \tan\theta = 4 \Rightarrow \tan^2\theta - 4\tan\theta + 1 = 0$$

$$\Rightarrow (\tan\theta - 2)^2 - 3 = 0 \Rightarrow (\tan\theta - 2 + \sqrt{3})(\tan\theta - 2 - \sqrt{3}) = 0 \Rightarrow \tan\theta = 2 - \sqrt{3} \text{ or } \tan\theta = 2 + \sqrt{3} \Rightarrow \theta = \frac{\pi}{12} \text{ or } \theta = \frac{5\pi}{12}$$

$\because\ \theta \in \left(0, \frac{\pi}{2}\right)$. Option C and D is correct.

28. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

(D) The foci of ellipse are $(\pm\sqrt{2}, 0)$

Sol. **(A,B)** $2x^2 - 2y^2 = 1$

$$\frac{x^2}{\left(\frac{1}{2}\right)} - \frac{y^2}{\left(\frac{1}{2}\right)} = 1 \qquad .....(i)$$

Eccentricity of hyperbola $= \sqrt{2}$

So eccentricity of ellipse $= \frac{1}{\sqrt{2}}$

Let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (a > b)$ $\qquad \Rightarrow \frac{1}{\sqrt{2}} = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow a^2 = 2b^2$

$\therefore\ x^2 + 2y^2 = 2b^2$ .....(ii)

Let ellipse and hyperbola intersect at $A\left(\frac{1}{\sqrt{2}}\sec\theta, \frac{1}{\sqrt{2}}\tan\theta\right)$.

Now, $4x - 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y} \Rightarrow \left.\frac{dy}{dx}\right|_{\text{at } A} = \frac{\sec\theta}{\tan\theta} = \operatorname{cosec}\theta$ $\qquad$ and $2x + 4y\frac{dy}{dx} = 0$

$$\left.\frac{dy}{dx}\right|_{\text{at } A} = -\frac{x}{2y} = -\frac{1}{2}\operatorname{cosec}\theta$$

Ellipse and hyperbola are orthogonal, so $\quad -\frac{1}{2}\operatorname{cosec}^2\theta = -1 \Rightarrow \operatorname{cosec}^2\theta = 2$

$$Q = \pm\frac{\pi}{4}$$

$$\therefore\ A\left(1, \frac{1}{\sqrt{2}}\right) \text{ or } \left(1, -\frac{1}{\sqrt{2}}\right)$$

$$1 + 2\left(\frac{1}{\sqrt{2}}\right)^2 = 2b^2 \Rightarrow b^2 = 1$$

Equation of ellipse is $x^2 + 2y^2 = 2$

Co-ordinate of foci $(\pm ae, 0) = \left(\pm\sqrt{2}\,.\,\frac{1}{\sqrt{2}}, 0\right) = (\pm 1, 0)$

Option A and B are correct.

Note : If major axis is along y-axis, then $\frac{1}{\sqrt{2}} = \sqrt{1 - \frac{a^2}{b^2}} \Rightarrow b^2 = 2a^2$

$$2x^2 + y^2 = 2a^2$$

$$y' = -\frac{2x}{y}$$

$$-\frac{2}{\sin\theta}\,.\operatorname{cosec}\theta = -1 \Rightarrow \operatorname{cosec}^2\theta = 1 \Rightarrow \theta = \pm\frac{\pi}{4}$$

$$2x^2 + y^2 = 2a^2 \Rightarrow 2 + \frac{1}{2} = 2a^2 \Rightarrow a^2 = \frac{5}{4}$$

$2x^2 + y^2 = \frac{5}{2}$, corresponding foci are $(0, \pm 1)$.

SECTION – III
Matrix – Match Type

This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statements in **Column I** are labelled A, B, C and D, while the statements in **Column II** are labelled p, q, r, s and t. Any given statement in **Column I** can have correct matching with **ONE OR MORE** statement(s) in **Column II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

If the correct matches are A–p, s and t; B–q and r; C–p and q; and D–s and t; then the correct darkening of bubbles will look like the following :

	p	q	r	s	t
A	● p	○ q	○ r	● s	● t
B	○ p	● q	● r	○ s	○ t
C	● p	● q	○ r	○ s	○ t
D	○ p	○ q	○ r	● s	● t

29. Match the statements/expressions given in **Column I** with the values given in **Column II.**

	Column I		Column II
(A)	The number of solutions of the equation $x\,e^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p)	1
(B)	Value(s) of k for which the planes $kx + 4y + z = 0,$ $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	(q)	2
(C)	Value(s) of k for which $\lvert x-1\rvert + \lvert x-2\rvert + \lvert x+1\rvert + \lvert x+2\rvert = 4k$ has integer solution(s)	(r)	3
(D)	If $y' = y + 1$ and $y(0) = 1,$ then value(s) of $y\,(\ln 2)$	(s)	4
		(t)	5

Sol. **(A) p; (B) q,s; (C) q,r,s,t, (D) r**

(A) Let $f(x) = x\,e^{\sin x} - \cos x$

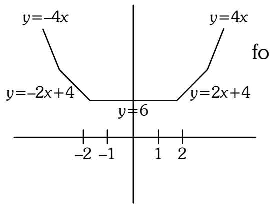
$f'(x) = e^{\sin x} + x\,e^{\sin x}\cos x + \sin x \ge 0$ for interval $x \in \left(0, \frac{\pi}{2}\right)$ $\Rightarrow f$ is strictly increasing

$f(0) = -1$

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}e \;\Rightarrow$ one solution.

(B) $\begin{vmatrix} k & 4 & 1 \\ 4 & k & 2 \\ 2 & 2 & 1 \end{vmatrix} = 0 \;\Rightarrow\; k(k-4) - 4\,(0) + 1\,(8-2k) = 0 \Rightarrow k^2 - 6k + 8 = 0 \;\Rightarrow\; k = 2, 4\,.$

(C)



for solutions, $4k \ge 6 \;\Rightarrow\; k \ge \frac{3}{2}\,.$

Integer values for k are 2, 3, 4, 5.

(D) $\frac{dy}{dx} = y + 1$

$\ln\lvert (y+1)\rvert = x + c$

$\ln 2 = c \;\Rightarrow\; \ln\lvert y+1\rvert = x + \ln 2$

put $x = \ln 2$

$\ln(y+1) = \ln 2 + \ln 2 = \ln 4 \Rightarrow\; y + 1 = 4 \;\Rightarrow\; y = 3\,.$

30. Match the statements/expressions given in **Column I** with the values given in **Column II.**

	Column I		**Column II**
(A)	Root(s) of the equation $2\sin^2\theta + \sin^2 2\theta = 2$	(p)	$\frac{\pi}{6}$
(B)	Points of discontinuity of the function $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$, where $[y]$ denotes the largest integer less than or equal to y	(q)	$\frac{\pi}{4}$
(C)	Volume of the parallelopiped with its edges represented by the vectors $\hat{i}+\hat{j},\ \hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$	(r)	$\frac{\pi}{3}$
(D)	Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a}+\vec{b}+\sqrt{3}\,\vec{c} = \vec{0}$	(s)	$\frac{\pi}{2}$
		(t)	π

Sol. **(A) q,s; (B) p, r, s, t; (C) t; (D) r**

(A) $2\sin^2\theta + \sin^2 2\theta = 2$

$$\Rightarrow \sin^2 2\theta = 2\cos^2\theta \Rightarrow 4\sin^2\theta\cos^2\theta = 2\cos^2\theta \quad \Rightarrow \cos^2\theta = 0 \text{ or } \sin^2\theta = \frac{1}{2} \Rightarrow \cos\theta = 0 \text{ or } \sin\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{\pi}{2}$$

(B) $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$

Possible points of discontinuity of $\left[\frac{6x}{\pi}\right]$ are $\frac{6x}{\pi} = n,\ n \in I \quad \Rightarrow x = \frac{n\pi}{6} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$

$$\lim_{x\to\frac{\pi}{6}^-} f(x) = 0\cos 0 = 0$$

$$\lim_{x\to\frac{\pi}{6}^+} f(x) = 1\cos 0 = 1$$

$\therefore$ Discontinuous at $x = \frac{\pi}{6}$. Similarly discontinuous at $x = \frac{\pi}{3}, \frac{\pi}{2}, \pi$

(C) $V = \left\|\begin{matrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{matrix}\right\| = \pi$ cubic units.

(D) $\vec{a}+\vec{b}+\sqrt{3}\,\vec{c} = \vec{0} \Rightarrow \vec{a}+\vec{b} = -\sqrt{3}\,\vec{c} \Rightarrow \left|\vec{a}+\vec{b}\right|^2 = \left|\sqrt{3}\,\vec{c}\right|^2$

$$\Rightarrow a^2 + b^2 + 2\vec{a}\,.\,\vec{b} = 3c^2 \Rightarrow 2 + 2\cos\theta = 3 \Rightarrow \cos\theta = \frac{1}{2}.$$

SECTION – IV
Integer Answer Type

This section contains 8 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following :

X	Y	Z	W
⓪	⓿	⓪	⓪
①	①	①	①
②	②	②	❷
③	③	③	③
④	④	④	④
⑤	⑤	⑤	⑤
❻	⑥	⑥	⑥
⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧
⑨	⑨	❾	⑨

31. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$. Then the value of $p(2)$ is

Sol. $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

$p'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$p'(1) = 4a + 3b + 2c + d = 0$ .....(i)

$p'(2) = 32a + 12b + 4c + d = 0$ .....(ii)

$$\lim_{x\to\infty}\left(1+\frac{p(x)}{x^2}\right) = 2$$

$$\lim_{x\to\infty}\frac{ax^4 + bx^3 + (c+1)x^2 + dx + e}{x^2} = 2$$

$c + 1 = 2, \quad d = 0, \quad e = 0 \qquad \Rightarrow c = 1\,,\ d = 0\,,\ e = 0$

Now equation (i) and (ii) are $4a + 3b = -2$ and $32a + 12b = -4 \Rightarrow a = \dfrac{1}{4}$ and $b = -1$.

32. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let *P* be the mid point of the line segment joining the centres of C_1 and C_2 and *C* be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and *C* passing through *P* is also a common tangent to C_2 and *C*, then the radius of the circle *C* is

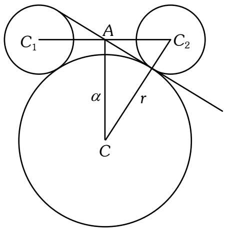
Sol. $(r+1)^2 = \alpha^2 + 9$

and $r^2 + 8 = \alpha^2$

$\Rightarrow r^2 + 2r + 1 = r^2 + 8 + 9$

$\Rightarrow 2r = 16$

$\Rightarrow r = 8$



33. Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function which satisfies $f(x) = \int_0^x f(t)\,dt$. Then the value of $f(\ln 5)$ is

Sol. From given integral equation, $f(0) = 0$.

Also, differentiating the given integral equation w.r.t. *x*

$f'(x) = f(x)$

If $f(x) \neq 0 \Rightarrow \dfrac{f'(x)}{f(x)} = 1 \Rightarrow f(x) = e^c e^x$

$\because\ f(0) = 0 \Rightarrow e^c = 0,$ a contradiction

$\therefore\ f(x) = 0\ \forall\ x \in R \quad \Rightarrow f(\ln 5) = 0$.

34. The smallest value of *k*, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is

Sol. (i) $x^2 - 8kx + 16(k^2 - k + 1) = 0$

$D = 64(k^2 - (k^2 - k + 1)) = 64(k - 1) > 0$

$k > 1$

(ii) $-\dfrac{b}{2a} > 4 \Rightarrow \dfrac{8k}{2} > 4 \Rightarrow k > 1$

(iii) $f(4) \geq 0$

$16 - 32k + 16(k^2 - k + 1) \geq 0 \Rightarrow k^2 - 3k + 2 \geq 0 \Rightarrow (k-2)(k-1) \geq 0 \Rightarrow k \leq 1$ or $k \geq 2$

Hence, $k = 2$.

35. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \left\{x \mid x^2 + 20 \leq 9x\right\}$ is

Sol. $A = \{x \mid x^2 + 20 \leq 9x\} = \{x \mid x \in [4, 5]\}$

Now, $f'(x) = 6(x^2 - 5x + 6)$

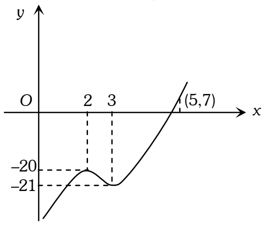
$f'(x) = 0$

$\Rightarrow x = 2, 3$

$f(2) = -20, \quad f(3) = -21$,

$f(4) = -16, \quad f(5) = 7$

From graph, maximum of $f(x)$ on set *A* is $f(5) = 7$.



36. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

Sol. $g(f(x)) = x$

$\Rightarrow g'(f(x))\,f'(x) = 1$ .....(i)

If $f(x) = 1 \Rightarrow x = 0, f(0) = 1$

Substitute $x = 0$ in (i), we get $g'(1) = \frac{1}{f'(0)}$ $\Rightarrow g'(1) = 2$. $\left[f'(x) = 3x^2 + \frac{1}{2}e^{x/2} \Rightarrow f'(0) = \frac{1}{2}\right]$

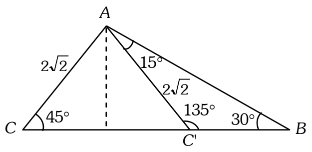
37. Let *ABC* and *ABC'* be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30°$. The absolute value of the difference between the areas of these triangles is

Sol. In ΔABC, by sine rule

$$\frac{a}{\sin A} = \frac{2\sqrt{2}}{\sin 30°} = \frac{4}{\sin C} \Rightarrow C = 45°,\ C' = 135°$$

When $C = 45° \Rightarrow A = 180° - (45° + 30°) = 105°$

When $C' = 135° \Rightarrow A = 180° - (135° + 30°) = 15°$

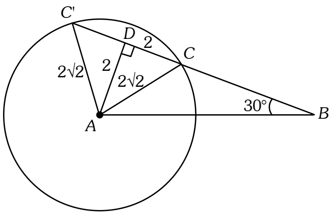


Area of $\Delta ABC' = \frac{1}{2} AB \times AC' \sin A = \frac{1}{2} \times 4 \times 2\sqrt{2}\ \sin(105°) = 4\sqrt{2} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$

Area of $\Delta ABC = \frac{1}{2} AB \times AC \sin A = \frac{1}{2} \times 4 \times 2\sqrt{2}\ \sin(15°) = 2(\sqrt{3} - 1)$

Difference of areas of triangles $= |\,2(\sqrt{3}+1) - 2(\sqrt{3}-1)\,| = 4$.

Aliter :



$AD = 2$, $DC = 2$

Difference of areas of triangle *ABC* and *ABC'* = Area of triangle *ACC'* $= \frac{1}{2} AD \times CC' = \frac{1}{2} \times 2 \times 4 = 4$.

38. Let (x, y, z) be points with integer coordinates satisfying the system homogeneous equations :

$3x - y - z = 0$

$-3x + z = 0$

$-3x + 2y + z = 0$. Then the number of such points for which $x^2 + y^2 + z^2 \le 100$ is

Sol. $3x - y - z = 0$ .....(i)

$-3x + 2y + z = 0$ .....(ii)

$-3x + z = 0$ .....(iii)

(i) + (ii), $y = 0$

So, $3x = z$

Now, $x^2 + y^2 + z^2 \le 100 \Rightarrow x^2 + (3x)^2 + 0 \le 100 \Rightarrow 10x^2 \le 100 \Rightarrow x^2 \le 10$

$x = -3, -2, -1, 0, 1, 2, 3$

So, number of such 7 points are possible.